Parton construction of Topologically Ordered Phases / Quantum Spin-liquids.

As we saw earlier, topologically ordered phases have excitations that carry fractional statistics e.g. Toric code has emergent ~~self~~fermion which has anyonic statistics with other quasiparticles. One way to convert a Hamiltonian written in terms of microscopic variables (e.g. spins/bosons) into a Hamiltonian written in terms of emergent excitations is the 'parton construction'.

e.g. consider $H = \sum_{\langle ij \rangle} J \, \vec{S}_i \cdot \vec{S}_j$ on some lattice. One can represent $\vec{S}$ as

$$\vec{S} = \begin{bmatrix} f^\dagger_\uparrow & f^\dagger_\downarrow \end{bmatrix} \frac{\vec{\sigma}}{2} \begin{bmatrix} f_\uparrow \\ f_\downarrow \end{bmatrix}$$ or

$$\vec{S} = \begin{bmatrix} b^\dagger_\uparrow & b^\dagger_\downarrow \end{bmatrix} \frac{\vec{\sigma}}{2} \begin{bmatrix} b_\uparrow \\ b_\downarrow \end{bmatrix}$$ where

f/b are ~~boso~~ fermion/boson operators. To implement $\vec{S}^2 = \frac{1}{2}\left(\frac{1}{2}+1\right)$, one requires

$$f^\dagger f = f^\dagger_\uparrow f_\uparrow + f^\dagger_\downarrow f_\downarrow = 1 = b^\dagger b.$$

In terms of f/b, H becomes,

$$H = \frac{J}{2} f^\dagger_{i\alpha} f_{j\alpha} f_{i\beta} f^\dagger_{j\beta} + \text{const.}$$

or

$$H = \frac{J}{2} b^\dagger_{i\alpha} b_{j\alpha} b_{i\beta} b^\dagger_{j\beta} + \text{const.}$$

So far, everything is exact.

One can now do a mean-field decomposition,

$$H \simeq \frac{J}{2} \Big[f^\dagger_{i\alpha} f_{j\alpha} \langle f_{i\beta} f^\dagger_{j\beta} \rangle + f^\dagger_{i\alpha} f^\dagger_{j\beta} \langle f_{j\alpha} f_{i\beta} \rangle \cancel{+ \text{const.}} \Big]$$

Whether this is a good ansatz or not will depend on the actual Hamiltonian, but <u>if</u> it is, then H corresponds to a $\mathbb{Z}_2$ guage theory,

$$H \simeq \frac{J}{2} \Big[t f^\dagger_{i\alpha} Z_{ij} f_{j\alpha} + \Delta_{\alpha\beta} f^\dagger_{i\alpha} Z_{ij} f^\dagger_{j\beta} + \ldots \Big]$$

$$t = \langle f_{i\beta} f^\dagger_{j\beta} \rangle \qquad \Delta_{\alpha\beta} = \langle f_{j\alpha} f_{i\beta} \rangle$$

The physical origin of the $\mathbb{Z}_2$ guage structure is the redundancy in the representation $\vec{S} = f^{+} \frac{\vec{\sigma}}{2} f$ or $\vec{S} = b^{+} \frac{\vec{\sigma}}{2} b$.

$f_i \to -f_i$ does not change $\vec{S}$, and this is a <u>local</u> redundancy, i.e. can be implemented locally on each site independently.

One way to obtain an approximate ground state for such a guage theory is to first neglect the guage fluctuations and solve the mean-field Hamiltonian.

$$H_{MF} \sim -t \left[f^{+}_{i\alpha} f_{j\beta} + h.c. \right] + \Delta_{\alpha\beta} \left[f^{+}_{i\alpha} f^{+}_{j\beta} + h.c. \right]$$

and then project the ground state to the physical Hilbert space that satisfies $f^{+} f = 1$.

i.e. $|\psi\rangle_{actual} = P_{f^{+}f=1} |\varphi_{MF}\rangle$

where $P_{f^{+}f=1}$ projects onto $f^{+}_i f_i = 1$

$\forall i$

The mean-field state $|\psi_{MF}\rangle$ corresponds to a superconductor and can be written as

$$|\psi\rangle_{MF} = \prod_k \left[1 + \frac{v_k}{u_k} f^\dagger_{k\uparrow} f^\dagger_{-k\downarrow}\right]|0\rangle$$

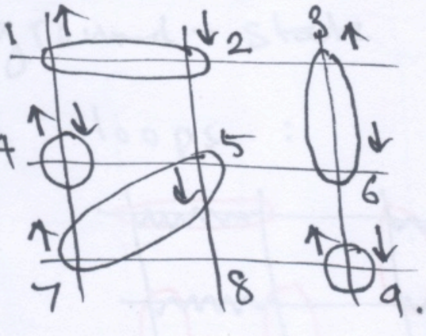
where v_k, u_k are f^ns of t_{ij}, $\Delta_{ij}^{\alpha f}$.

Projecting onto a fixed particle number space ($\sum_i f^\dagger_i f_i = N$),

$$|\psi_{MF}\rangle = \left[\sum_{ij} g(r_i - r_j) f^\dagger_{i\uparrow} f^\dagger_{j\downarrow}\right]^{N/2} |0\rangle$$

where g = Fourier transform of $\frac{v_k}{u_k}$.

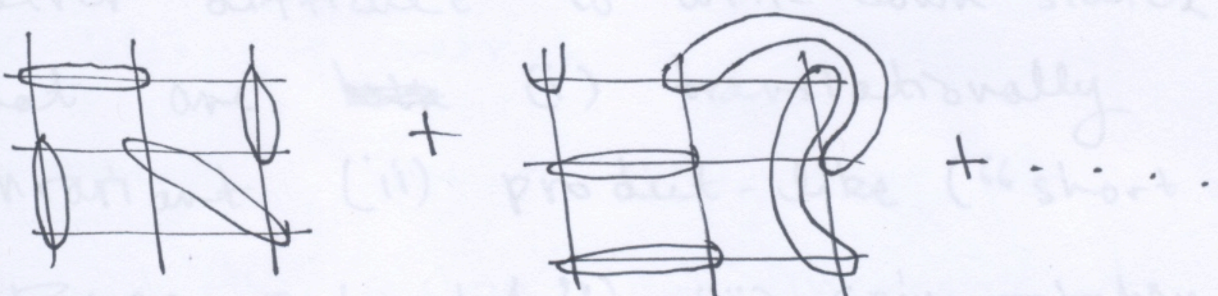
Thus the mean-field state can be written as superposition of states where each state is a product state of cooper-pairs.



$$= g_{12}\ g_{36}\ g_{44}\ g_{75}\ g_{99} \ldots$$

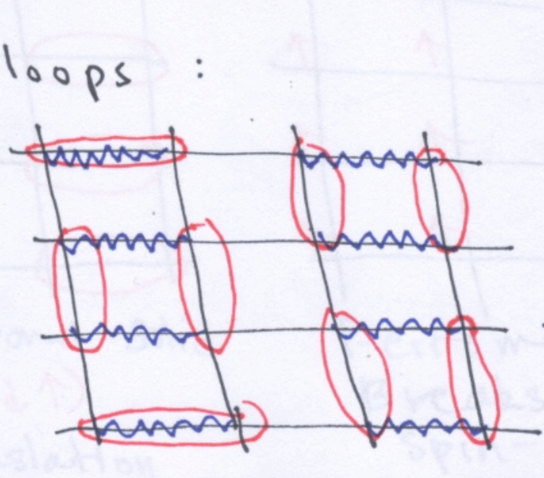
(up to a sign).

The projection $f^\dagger f = 1$ removes states that have either no f-fermions on a given site, or two f-fermions. For example, the above state is projected out. After projection, the ground state resembles a superposition of 'dimers' where each dimer corresponds to a cooper pair.



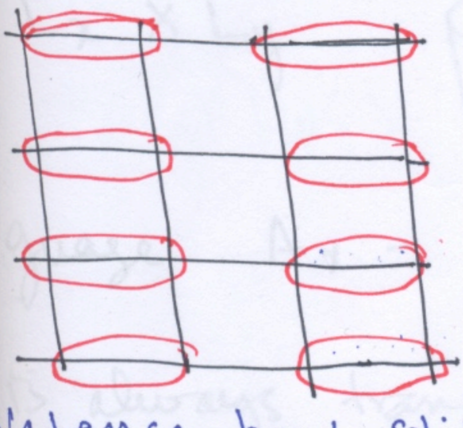
This is the 'resonating valence bond' (RVB) picture of spin-liquids.

For short-range dimers (cooper-pairs), the ground-state resembles sum of closed loops:



The blue wiggles denote a reference dimer config. consistent with $f^\dagger f = 1$. The overlap between red and blue yields closed loops.

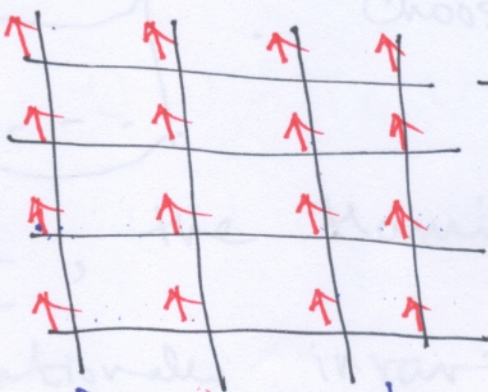
The parton construction provides a recipe to construct translationally invariant w-f^n's for systems that have $f^{\dagger} f = 1$ i.e. one spin-$1/2$ per unit cell. As one can intuitively see that one (or in general an odd number) of spin-$1/2$ spins / unit cell makes it rather difficult to write down states that are (i) translationally invariant (ii) product-like ("short-range entangled"). (iii) spin-rotation invariant. e.g. ~~the~~ consider the three states:



Valence bond solid

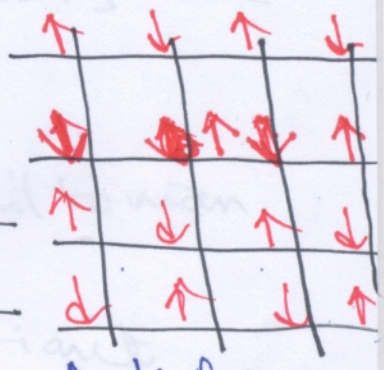
$\bigcirc = |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$

Breaks translation



Ferromagnet

Breaks spin-rotation



Anti-ferro

Breaks translation and spin-rotation.

There is a general argument by Oshikawa, later generalized by Hastings, that shows that a system with a U(1) sym. with U(1) charge ("filling") $= \frac{p}{q}$, must have atleast q ground states. This rules out the presence of a unique g.s. that is U(1) symmetric.

<u>Argument</u>: One inserts flux Φ through the system and increase Φ from 0 to 2π. ~~Choosi~~. The system is put on cylinder with dimensions $L_x \times L_y$

Choosing the guage $A_x = \frac{\Phi}{L}$, the Hamiltonian is always translationally invariant and therefore the eigenvalue of the ~~mom~~ translation operator T_x doesn't change during the process.

If the initial g.s. (i.e. at $\phi = 0$) satisfies $T_x |\psi_0\rangle = e^{iPx} |\psi_0\rangle$, then the ground state at $\phi = 2\pi$, $|\psi'\rangle$ also satisfies $T_x |\psi'\rangle = e^{iPx} |\psi'\rangle$.

One can find a unitary operator U s.t. $U^{+} H(\phi = 2\pi) U = H(\phi = 0)$

U is given by $U = e^{\frac{2\pi i}{L_x} \sum_{\vec{r}} x \, n_{\vec{r}}}$. Since U brings back H to the original $H = H(\phi=0)$, one can now compare the momentum of $|\psi_0\rangle$ with $U |\psi'_0\rangle$:

$$T_x U |\psi'_0\rangle = U U^{-1} T_x U |\psi'_0\rangle$$

One can show that $U^{-1} T_x U = T_x e^{i \frac{2\pi}{L} \hat{N}}$

where $\hat{N} = \sum_{\vec{r}} n_r$ is the operator corresponding to total particle number.

$\Rightarrow T_x U|\psi'_0\rangle = U T_x e^{\frac{2\pi i N}{L_x}} |\psi'_0\rangle$

$= U e^{\frac{2\pi i N}{L} + i p^0_x} |\psi'_0\rangle$

$\Rightarrow$ the momentum of the final state

$= \frac{2\pi N}{L_x} + p^0_x$

$N = \frac{p}{q} L_x L_y$

$\Rightarrow$ final momentum $= p^0_x + 2\pi \frac{p}{q} L_y$

Choosing L_y and q coprime,

change in momentum $= 2\pi \frac{p L_y}{q} \neq 2\pi \times$ integer

$\Rightarrow$ final state is orthogonal to the initial state while having same eigenvalue of H $\Rightarrow$ the two states are degenerate, orthogonal g.s. One can repete the argument $q-1$ times to obtain

$q-1$ states orthogonal and degenerate with $U|\psi'_0\rangle$.

$\Rightarrow$ atleast q ground states at filling p/q.

This degeneracy may arise due to spontaneous sym. breaking or topological order. the theorem is too general to tell anything about how it appears.
